

# A discrete form of the Beckman-Quarles theorem for two-dimensional strictly convex normed spaces

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version of October 8, 2000

## Abstract

Let  $X$  be a real normed vector space and  $\dim X \geq 2$ . Let  $\rho > 0$  be a fixed real number. We prove that if  $x, y \in X$  and  $\|x - y\|/\rho$  is a rational number then there exists a finite set  $\{x, y\} \subseteq S_{xy} \subseteq X$  with the following property: for each strictly convex  $Y$  of dimension 2 each map from  $S_{xy}$  to  $Y$  preserving the distance  $\rho$  preserves the distance between  $x$  and  $y$ . It implies that each map from  $X$  to  $Y$  that preserves the distance  $\rho$  is an isometry.

Let  $\mathbf{Q}$  denote the field of rational numbers. All vector spaces mentioned in this article are assumed to be real. A normed vector space  $E$  is called *strictly convex* ([5]), if for each pair  $a, b$  of nonzero elements in  $E$  such that  $\|a + b\| = \|a\| + \|b\|$ , it follows that  $a = \gamma b$  for some  $\gamma > 0$ . It is known ([15]) that two-dimensional strictly convex normed spaces satisfy the following condition (\*):

(\*) for any  $a \neq b$  on line  $L$  and any  $c, d$  on the same side of  $L$ , if  $\|a - c\| = \|a - d\|$  and  $\|b - c\| = \|b - d\|$ , then  $c = d$ .

Conversely ([15]), for any two-dimensional normed space the condition (\*) implies that the space is strictly convex.

The classical Beckman-Quarles theorem states that any map from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  ( $2 \leq n < \infty$ ) preserving unit distance is an isometry, see [1], [2] and [6]. Various unanswered questions and counterexamples concerning the Beckman-Quarles theorem and isometries are discussed by Ciesielski and Rassias [4]. For more open problems and new results on isometric mappings the reader is referred to [7]-[13]. The Theorem below may be viewed as a discrete form of the Beckman-Quarles theorem for two-dimensional strictly convex normed spaces.

**Theorem.** Let  $X$  and  $Y$  be normed vector spaces such that  $\dim X \geq \dim Y = 2$  and  $Y$  is strictly convex. Let  $\rho > 0$  be a fixed real number.

1. If  $x, y \in X$  and  $\|x - y\|/\rho$  is a rational number then there exists a finite set  $S_{xy} \subseteq X$  containing  $x$  and  $y$  such that each injective map  $f : S_{xy} \rightarrow Y$  preserving the distance  $\rho$  preserves the distance between  $x$  and  $y$ .
2. If  $x, y \in X$  and  $\varepsilon > 0$  then there exists a finite set  $T_{xy}(\varepsilon) \subseteq X$  containing  $x$  and  $y$  such that each injective map  $f : T_{xy}(\varepsilon) \rightarrow Y$  preserving the distance  $\rho$  preserves the distance between  $x$  and  $y$  to within  $\varepsilon$  i.e.

$$|||f(x) - f(y)|| - \|x - y|| \leq \varepsilon.$$

3. Let  $X = \mathbb{R}^n$  ( $2 \leq n < \infty$ ) be equipped with euclidean norm. Then the assumption of injectivity is unnecessary in items 1 and 2.
4. More generally (cf. item 3), for each normed space  $X$  the assumption of injectivity is unnecessary in items 1 and 2.

**Proof of item 1.** Let  $D$  denote the set of all non-negative numbers  $d$  with the following property (\*\*):

- (\*\*) if  $x, y \in X$  and  $\|x - y\| = d$  then there exists a finite set  $S_{xy} \subseteq X$  such that  $x, y \in S_{xy}$  and any injective map  $f : S_{xy} \rightarrow Y$  that preserves the distance  $\rho$  also preserves the distance between  $x$  and  $y$ .

Obviously  $0, \rho \in D$ . We first prove that if  $d \in D$ , then  $2 \cdot d \in D$ . Assume that  $d \in D$ ,  $d > 0$ ,  $x, y \in X$ ,  $\|x - y\| = 2 \cdot d$ . Using the notation of Figure 1

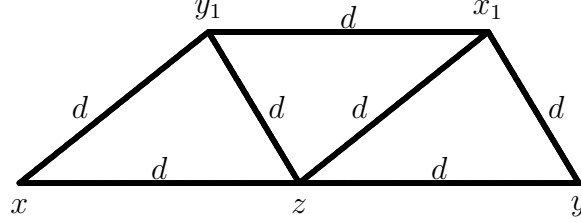


Figure 1

$$\|x - y\| = 2 \cdot d$$

$$z := \frac{x+y}{2}$$

$$\|x - z\| = \|x - y_1\| = \|z - y_1\| = d$$

$$x_1 := y_1 + (z - x)$$

we show that

$$S_{xy} := S_{xz} \cup S_{zy} \cup S_{y_1x_1} \cup S_{xy_1} \cup S_{zx_1} \cup S_{zy_1} \cup S_{yx_1}$$

satisfies the condition (\*\*). Let an injective  $f : S_{xy} \rightarrow Y$  preserves the distance  $\rho$ . By the injectivity of  $f$ :  $f(x) \neq f(x_1)$  and  $f(y) \neq f(y_1)$ . According to (\*):  $f(y_1) - f(x_1) = f(x) - f(z)$  and  $f(y_1) - f(x_1) = f(z) - f(y)$ . Hence  $f(x) - f(z) = f(z) - f(y)$ . Therefore  $\|f(x) - f(y)\| = \|2(f(x) - f(z))\| = 2 \cdot \|f(x) - f(z)\| = 2 \cdot \|x - z\| = 2 \cdot d = \|x - y\|$ .

From Figure 2, the previous step and the property that defines strictly convex normed spaces it is clear that if  $d \in D$ , then all distances  $k \cdot d$  ( $k$  a positive integer) belong to  $D$ .

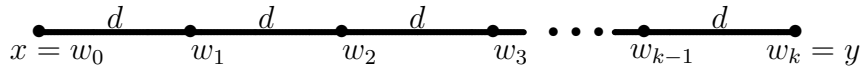


Figure 2

$$\|x - y\| = k \cdot d$$

$$S_{xy} = \bigcup \{S_{ab} : a, b \in \{w_0, w_1, \dots, w_k\}, \|a - b\| = d \vee \|a - b\| = 2 \cdot d\}$$

From Figure 3, the previous step and the property that defines strictly convex normed spaces it is clear that if  $d \in D$ , then all distances  $d/k$  ( $k$  a

positive integer) belong to  $D$ . Hence  $D/\rho := \{d/\rho : d \in D\} \supseteq \mathbf{Q} \cap [0, \infty)$ . This completes the proof of item 1.

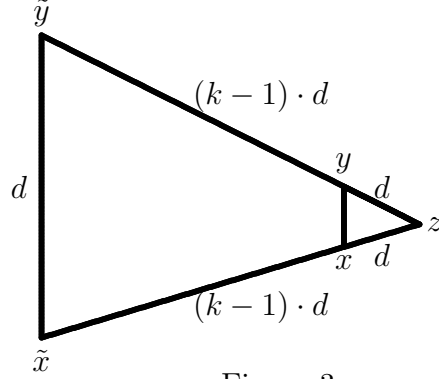


Figure 3

$$||x - y|| = d/k$$

$$\tilde{x} := x + (k-1)(x-z)$$

$$\tilde{y} := y + (k-1)(y-z)$$

$$\tilde{x} - \tilde{y} = x - y + (k-1)((x-z) - (y-z)) = k(x-y)$$

$$S_{xy} = S_{\tilde{x}\tilde{y}} \cup S_{\tilde{x}x} \cup S_{xz} \cup S_{\tilde{x}z} \cup S_{\tilde{y}y} \cup S_{yz} \cup S_{\tilde{y}z}$$

**Proof of item 2.** It follows from Figure 4.



Figure 4

$$|x-z|/\rho, |z-y|/\rho \in \mathbf{Q} \cap [0, \infty), |z-y| \leq \varepsilon/2$$

$$T_{xy}(\varepsilon) = S_{xz} \cup S_{zy}$$

**Proof of item 3.** In proofs of items 1 and 2 the assumption of injectivity is necessary only in the first step for distances  $2 \cdot d$ ,  $d \in D$ . Let  $X = \mathbb{R}^n$  ( $2 \leq n < \infty$ ) be equipped with euclidean norm and  $D$  is defined without the assumption of injectivity. Let  $d \in D$ ,  $d > 0$ . We need to prove that  $2 \cdot d \in D$ . Let us see at configuration from Figure 5 below, all segments have the length  $d$ .

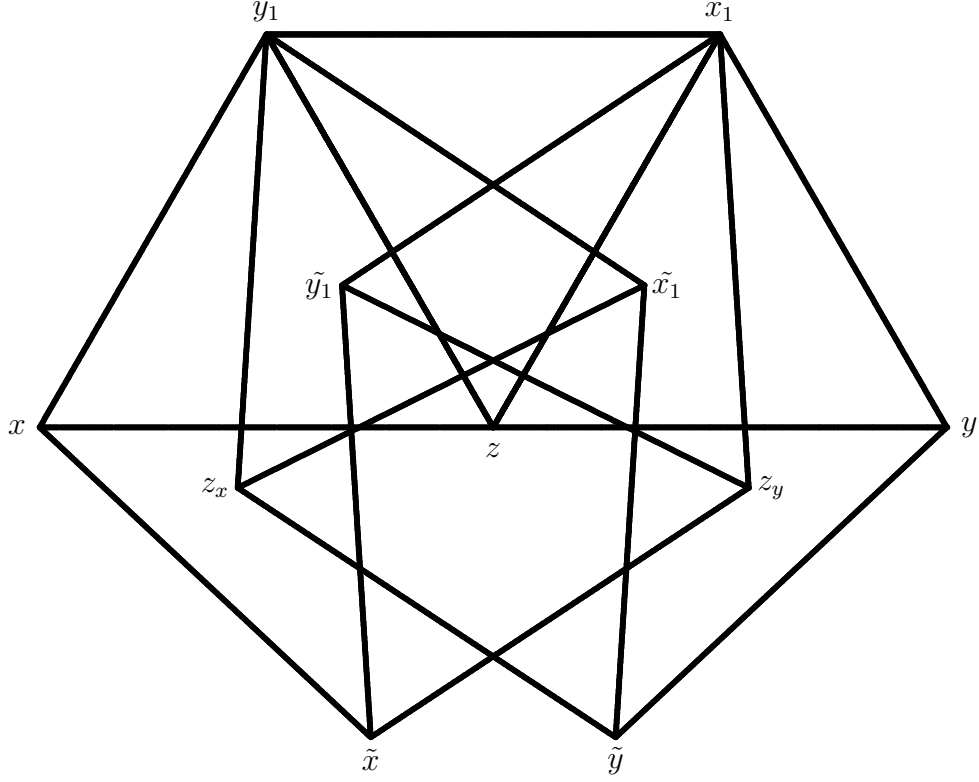


Figure 5

$$\|x - y\| = 2 \cdot d$$

$$z := \frac{x+y}{2}$$

$$S_{xy} = \cup \{S_{ab} : a, b \in \{x, \tilde{x}, x_1, \tilde{x}_1, y, \tilde{y}, y_1, \tilde{y}_1, z, z_x, z_y\}, \|a - b\| = d\}$$

Assume that  $f : S_{xy} \rightarrow Y$  preserves the distance  $\rho$ . It is sufficient to prove that  $f(x) \neq f(x_1)$  and similarly  $f(y) \neq f(y_1)$ . Suppose, on the contrary, that  $f(x) = f(x_1)$ , the proof of  $f(y) \neq f(y_1)$  is similar. Hence four points:  $f(\tilde{x})$ ,  $f(z_y)$ ,  $f(\tilde{y}_1)$ ,  $f(x_1)$  have the distance  $d$  from each other. We prove that it is impossible in two-dimensional strictly convex normed spaces. Suppose, on the contrary, that  $a_1, a_2, a_3, a_4 \in Y$  and  $\|a_1 - a_2\| = \|a_1 - a_3\| = \|a_1 - a_4\| = \|a_2 - a_3\| = \|a_2 - a_4\| = \|a_3 - a_4\| = d > 0$ . Let us consider the segment

$a_2a_3$ . According to  $(*)$   $a_1$  and  $a_4$  lie on the opposite sides of the line  $L(a_2, a_3)$  and  $a_2 - a_1 = a_4 - a_3$ . Let us consider the segment  $a_1a_3$ . According to  $(*)$   $a_2$  and  $a_4$  lie on the opposite sides of the line  $L(a_1, a_3)$  and  $a_1 - a_2 = a_4 - a_3$ . Hence  $a_4 - a_3 = 0$ , a contradiction. This completes the proof of item 3.

**Proof of item 4.** Analogously as in the proof of item 3 it suffices to prove that for each  $x, y \in X$ ,  $x \neq y$  there exist points forming the configuration from Figure 5 where all segments have the length  $\|x - y\|/2$ . Let us consider  $x, y \in X$ ,  $x \neq y$ . We choose two-dimensional subspace  $\tilde{X} \subseteq X$  containing  $x$  and  $y$ .

**First case:** the norm induced on  $\tilde{X}$  is strictly convex. Obviously  $\tilde{X}$  is isomorphic to  $\mathbb{R}^2$  as a linear space. Let us consider  $\mathbb{R}^2$  with a strictly convex norm  $\|\cdot\|$ . It suffices to prove that for each  $a, b \in \mathbb{R}^2$  satisfying  $\|a\| = \|b\| = \|a - b\| = d > 0$  there exist  $\tilde{a}, \tilde{b} \in \mathbb{R}^2$  satisfying  $\|\tilde{a}\| = \|\tilde{b}\| = \|\tilde{a} - \tilde{b}\| = \|(\tilde{a} + \tilde{b}) - (a + b)\| = d$ . We fix  $a = (a_x, a_y)$  and  $b = (b_x, b_y)$ . Let  $S := \{x \in \mathbb{R}^2 : \|x\| = d\}$ . According to  $(*)$  for each  $u = (u_x, u_y) \in S$  there exists a unique  $h(u) = (h(u)_x, h(u)_y) \in S$  such that  $\|u - h(u)\| = d$  and

$$\det \begin{bmatrix} u_x & u_y \\ h(u)_x & h(u)_y \end{bmatrix} \cdot \det \begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix} > 0.$$

Obviously  $h(a) = b$ . The mapping  $h : S \rightarrow S$  is continuous. For each  $u \in S$   $h(-u) = -h(u)$  and  $\|u + h(u)\| = \|2u - (u - h(u))\| \geq \| \|2u\| - \|u - h(u)\| \| = d$ . The following function

$$S \ni x \xrightarrow{g} \|x + h(x) - a - h(a)\| \in [0, \infty)$$

is continuous. We have:

$$g(a) = 0,$$

$$g(-a) = \| -a + h(-a) - a - h(a) \| = 2 \cdot \|a + h(a)\| \geq 2 \cdot d.$$

Since  $g$  is continuous there exists  $\tilde{a} \in S$  such that  $g(\tilde{a}) = d$ . From this  $\tilde{a}$  and  $\tilde{b} := h(\tilde{a})$  satisfy  $\|\tilde{a}\| = \|\tilde{b}\| = \|\tilde{a} - \tilde{b}\| = \|(\tilde{a} + \tilde{b}) - (a + b)\| = d$ . This

completes the proof of item 4 in the case where the norm induced on  $\tilde{X}$  is strictly convex.

**Second case:** we assume only that  $\| \cdot \|$  is a norm on  $\tilde{X}$ . The graph  $\Gamma$  from Figure 5 (11 vertices, 19 edges) has the following matrix representation:

	$v_0$	$v_1$	$v_2$	$v_3$	$v_4$	$v_5$	$v_6$	$v_7$	$v_8$	$v_9$	$v_{10}$
$v_0 := x$	0	0	1	1	0	0	0	1	0	0	0
$v_1 := y$	0	0	1	0	1	0	1	0	0	0	0
$v_2 := \frac{x+y}{2}$	1	1	0	0	1	0	0	1	0	0	0
$v_3 := \tilde{x}$	1	0	0	0	0	0	0	0	1	0	1
$v_4 := x_1$	0	1	1	0	0	0	0	1	1	0	1
$v_5 := \tilde{x}_1$	0	0	0	0	0	0	1	1	0	1	0
$v_6 := \tilde{y}$	0	1	0	0	0	1	0	0	0	1	0
$v_7 := y_1$	1	0	1	0	1	1	0	0	0	1	0
$v_8 := \tilde{y}_1$	0	0	0	1	1	0	0	0	0	0	1
$v_9 := z_x$	0	0	0	0	0	1	1	1	0	0	0
$v_{10} := z_y$	0	0	0	1	1	0	0	0	1	0	0

Let  $u_0 := v_0 = x$ ,  $u_1 := v_1 = y$ ,  $u_2 := v_2 = \frac{x+y}{2}$ . We define the following function  $\psi$ :

$$\tilde{X}^8 \ni (u_3, \dots, u_{10}) \xrightarrow{\psi} (\|u_i - u_j\| : 0 \leq i < j \leq 10, (v_i, v_j) \in \Gamma) \in \mathbb{R}^{19}.$$

The image of  $\psi$  is a closed subset of  $\mathbb{R}^{19}$ . For each  $\varepsilon > 0$  and each bounded  $B \subseteq \tilde{X}$  the norm  $\| \cdot \|$  may be approximate on  $B$  with  $\varepsilon$ -accuracy by a strictly convex norm on  $\tilde{X}$ . Therefore according to the first case for each  $x, y \in X$ ,  $x \neq y$  and each  $\varepsilon > 0$  there exist points forming the configuration from Figure 5 where all segments have  $\| \cdot \|$ -lengths belonging to the interval  $(\frac{\|x-y\|}{2} - \varepsilon, \frac{\|x-y\|}{2} + \varepsilon)$ . Therefore:

$$(\|x - y\|/2, \dots, \|x - y\|/2) \in \overline{\psi(\tilde{X}^8)} \text{ (the closure of } \psi(\tilde{X}^8)\text{)}.$$

Since  $\psi(\tilde{X}^8)$  is closed we conclude that

$$(\|x - y\|/2, \dots, \|x - y\|/2) \in \psi(\tilde{X}^8).$$

This completes the proof of item 4.

**Corollary.** Let  $X$  and  $Y$  be normed vector spaces such that  $\dim X \geq \dim Y = 2$  and  $Y$  is strictly convex. From item 2 of the Theorem follows that an injective map  $f : X \rightarrow Y$  that preserves a fixed distance  $\rho > 0$  is an isometry. According to item 4 of the Theorem the assumption of injectivity is unnecessary in the above statement.

**Remark 1.** The set  $S_{xy}$  constructed in the proof does not depend on  $Y$ .

**Remark 2.** Instead of injectivity in the Theorem and Corollary we may assume that

$$\forall u, v \in \text{dom}(f) (\|u - v\|/\rho \in \mathbf{Q} \cap (0, \infty) \Rightarrow \|f(u) - f(v)\| \neq \|u - v\|/2)$$

It follows from Figure 1.

**Remark 3.** W. Benz and H. Berens proved ([3], see also [2] and [10]) the following theorem: Let  $X$  and  $Y$  be normed vector spaces such that  $Y$  is strictly convex and such that the dimension of  $X$  is at least 2. Let  $\rho > 0$  be a fixed real number and let  $N > 1$  be a fixed integer. Suppose that  $f : X \rightarrow Y$  is a mapping satisfying:

$$\begin{aligned} \|a - b\| = \rho &\Rightarrow \|f(a) - f(b)\| \leq \rho \\ \|a - b\| = N\rho &\Rightarrow \|f(a) - f(b)\| \geq N\rho \end{aligned}$$

for all  $a, b \in X$ . Then  $f$  is an affine isometry.

**Remark 4.** A. Tyszka proved ([14]) the following theorem: if  $x, y \in \mathbb{R}^n$  ( $2 \leq n < \infty$ ) and  $|x - y|$  is an algebraic number then there exists a finite set  $S_{xy} \subseteq \mathbb{R}^n$  containing  $x$  and  $y$  such that each map from  $S_{xy}$  to  $\mathbb{R}^n$  preserving unit distance preserves the distance between  $x$  and  $y$ .

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